

On the Aharonov-Bohm Hamiltonian

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Abstract

Using the theory of self-adjoint extensions, we construct all the possible hamiltonians describing the non relativistic Aharonov-Bohm effect. In general the resulting hamiltonians are not rotationally invariant so that the angular momentum is not a constant of motion. Using an explicit formula for the resolvent, we describe the spectrum and compute the generalized eigenfunctions and the scattering amplitude.

1. Introduction

In this paper we discuss the dynamics of a non relativistic, spinless quantum particle interacting with a magnetic field confined in a thin, infinite solenoid. Ignoring the irrelevant coordinate along the solenoid, the problem reduces to two dimensions and, if the radius of the solenoid goes to zero while the flux of the magnetic field is kept constant, one has a particle moving in R^2 subject to a δ -like magnetic field.

At a classical level such a particle has a trivial dynamics while the quantum mechanical description reveals a non trivial scattering cross section, explicitly depending on the flux of the magnetic field.

Then one has a purely quantum effect (the Aharonov-Bohm effect ([AB])) due to the non local character of the wave function, which is still debated in the literature both in its experimental and theoretical aspects (see e.g. [R] and references therein).

Here we shall concentrate on the mathematical problem of the construction of the hamiltonian describing the quantum dynamics.

Taking the origin of the coordinate system at the position of the solenoid and introducing the vector potential $A(x, y) = -\alpha c e^{-1}(-y(x^2 + y^2)^{-1}, x(x^2 + y^2)^{-1})$, where $-2\pi c \alpha e^{-1}$ is the magnetic flux through the solenoid, the formal hamiltonian of the particle written in polar coordinates reads

$$\hat{H}_\alpha = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(i \frac{\partial}{\partial \phi} - \alpha \right)^2. \quad (1.1)$$

In (1.1) we have fixed $\hbar = 1$, $m = 1/2$; moreover, without loss of generality, we suppose $0 < \alpha < 1$.

A natural starting point for the construction of a self-adjoint (s.a.) hamiltonian is to define (1.1) on a domain of smooth functions vanishing for $r = 0$, e.g. $C_0^\infty(R^2 \setminus \{0\})$, and then look for the possible s.a. extensions.

Each of the extensions will be characterized by a specific behaviour, i.e. boundary condition, of the elements of the domain near $r = 0$.

If one requires regularity in $r = 0$ for the elements of the domain of the extension, one obtains the hamiltonian studied in [AB].

Using the partial wave expansion, the hamiltonian can be reduced to each subspace of fixed angular momentum and a complete set of eigenfunctions can be constructed, i.e. the model is explicitly solvable.

In recent years it has been realized ([GHKL],[GMS],[MT]) that, if one drops

the assumption of regularity in $r = 0$, a new two-parameter group of s.a. extensions can be constructed.

Such extensions are obtained analysing (1.1) reduced separately to the subspaces of angular momentum zero (s-wave) and minus one (p-wave). These extensions obviously commute with angular momentum.

Explicit computation of the corresponding new bound states and scattering cross sections have also been given.

These hamiltonians should be interpreted as describing the magnetic interaction plus a contact interaction of the particle with the solenoid, in analogy with the contact or point interactions perturbing the free laplacian ([AGH-KH]).

The relevant difference is that in the latter case a point interaction can only be defined in the s-wave, while in our case the presence of the solenoid makes it possible to construct a point interaction also in the p-wave.

In this paper we generalize the previous results, developping further the work contained in [A].

We give a complete description of all the possible s.a. extensions of (1.1) defined on $C_0^\infty(R^2 \setminus \{0\})$.

We shall make use of the standard theory of s.a. extensions of Von Neumann and Krein (see e.g. [RSII],[AG]).

Since the deficiency indices are $(2, 2)$, there is a family of s.a. extensions of (1.1), parametrized by the unitary map U from one deficiency subspace to the other. Since the subspaces have two dimensions, the parametrization involves four real parameters.

The interesting point is that in general U realizes a coupling between the s- and p-waves and then H_α^U is not rotationally invariant, i.e. the angular momentum is not a constant of motion.

Only for special values of the parameters in U the previous, rotationally invariant extensions, are obtained.

To our knowledge, this new vorticity effect produced by the solenoid has not been discussed in the literature.

We shall also discuss the spectral properties of H_α^U . In particular we shall analyze the occurrence of bound states and we shall prove asymptotic completeness. Using the eigenfunction expansion we shall give rather explicit formulas for the scattering amplitude.

A final comment concerns an alternative description of the Aharonov-Bohm problem and its relation with the problem of anyons, i.e. particles in R^2 satisfying fractional statistics (see e.g. [W] and references therein).

It is well known that a unitary map reduces (1.1) to the free laplacian with boundary conditions on the positive x-axis. The existence of the dynamics for the pure magnetic interaction is then proved using the methods of potential theory. The construction generalizes to the case of N solenoids ([S],[DFT]), but in this approach it seems more difficult to introduce the richer structure produced by the contact interactions discussed here.

The hamiltonian (1.1) is also unitarily equivalent to the hamiltonian for the relative motion of two anyons. Our results can then be read as the construction of the most general hamiltonian for two anyons with point interaction, the new feature being that such point interaction is not necessarily rotationally invariant.

The general N-anyons problem with two-body point interactions seems to be difficult to control with the approach proposed here. A more natural setting is probably the one discussed in [DFT].

The remaining part of the paper is organized as follows.

In Section 2 we give the construction of H_α^U .

In Section 3 we derive an explicit formula for the resolvent and describe the spectrum.

In Section 4 we compute the generalized eigenfunctions and the scattering amplitude.

2. Self-adjoint extensions

As stated in Sec. 1, we start considering the symmetric and positive operator \dot{H}_α given by (1.1) with $D(\dot{H}_\alpha) = C_0^\infty(R^2 \setminus \{0\})$, in the Hilbert space $L^2(R^2)$. We denote its closure by \dot{H}_α , where

$$D(\dot{H}_\alpha) = \left\{ u \in L^2(R^2) \mid u \in H_{loc}^2(R^2 \setminus \{0\}), \dot{H}_\alpha u \in L^2(R^2) \right\}. \quad (2.1)$$

H^n (resp. H_{loc}^n) is the standard Sobolev space (resp. local Sobolev space) of order n .

To construct all the s.a. extensions of \dot{H}_α we consider the solution of the equation $\dot{H}_\alpha^* \psi = \pm i \psi$, where \dot{H}_α^* is the adjoint of \dot{H}_α .

This equation is more conveniently studied if we introduce the decomposition of the Hilbert space $L^2(R^2)$ with respect to angular momentum

$$L^2(R^2) = L^2(R^+; r dr) \otimes L^2(S^1), \quad (2.2)$$

where $R^+ = (0, \infty)$ and S^1 is the unit sphere in R^2 . Using the unitary transformation

$$V : L^2(R^+; r dr) \rightarrow L^2(R^+) \quad (Vf)(r) = r^{1/2}f(r) \quad (2.3)$$

and the completeness of $\frac{e^{im\phi}}{\sqrt{2\pi}}$, $m \in Z$, in $L^2(S^1)$, we can also write

$$L^2(R^2) = \bigoplus_{m=-\infty}^{+\infty} \left(V^{-1}L^2(R^+) \right) \otimes \left[\frac{e^{im\phi}}{\sqrt{2\pi}} \right], \quad (2.4)$$

where $\left[\frac{e^{im\phi}}{\sqrt{2\pi}} \right]$ is the linear span of $\frac{e^{im\phi}}{\sqrt{2\pi}}$. Corresponding to the decomposition (2.4) one has

$$\dot{H}_\alpha = \bigoplus_{m=-\infty}^{+\infty} V^{-1} \dot{h}_{\alpha,m} V \otimes 1 \quad (2.5)$$

where the operators $\dot{h}_{\alpha,m}$ in $L^2(R^+)$ are defined by (see e.g. [AGH-KH])

$$\begin{aligned} D(\dot{h}_{\alpha,0}) = & \left\{ \xi \in L^2(R^+) \mid \xi, \xi' \in H_{loc}^1(R^+), \right. \\ & \left. -\xi'' + (\alpha^2 - 1/4)r^{-2}\xi \in L^2(R^+), W(\xi, \xi_{\pm}^{(0)})_{0+} = 0 \right\}, \end{aligned} \quad (2.6)$$

$$\begin{aligned} D(\dot{h}_{\alpha,-1}) = & \left\{ \xi \in L^2(R^+) \mid \xi, \xi' \in H_{loc}^1(R^+), \right. \\ & \left. -\xi'' + [(1-\alpha)^2 - 1/4]r^{-2}\xi \in L^2(R^+), W(\xi, \xi_{\pm}^{(-1)})_{0+} = 0 \right\} \end{aligned} \quad (2.7)$$

$$\begin{aligned} D(\dot{h}_{\alpha,m}) = & \left\{ \xi \in L^2(R^+) \mid \xi, \xi' \in H_{loc}^1(R^+), \right. \\ & \left. -\xi'' + [(m+\alpha)^2 - 1/4]r^{-2}\xi \in L^2(R^+) \right\}, \quad m \neq 0, -1, \end{aligned} \quad (2.8)$$

$$\dot{h}_{\alpha,m}\xi = -\frac{d^2\xi}{dr^2} + \frac{(m+\alpha)^2 - 1/4}{r^2}\xi, \quad m \in Z \quad (2.9)$$

Here $W(f, g)_x = \overline{f(x)}g'(x) - \overline{f'(x)}g(x)$ denotes the Wronskian of f and g evaluated in x and

$$\xi_+^{(0)}(r) = Nr^{1/2}K_\alpha\left(e^{-i\frac{\pi}{4}}r\right), \quad \xi_-^{(0)}(r) = Ne^{i\frac{\pi}{2}\alpha}r^{1/2}K_\alpha\left(e^{i\frac{\pi}{4}}r\right), \quad (2.10)$$

$$\xi_+^{(-1)}(r) = Mr^{1/2}K_{1-\alpha}\left(e^{-i\frac{\pi}{4}}r\right), \quad \xi_-^{(-1)}(r) = Me^{i\frac{\pi}{2}(1-\alpha)}r^{1/2}K_{1-\alpha}\left(e^{i\frac{\pi}{4}}r\right), \quad (2.11)$$

$$N = \frac{\sqrt{2\cos\frac{\pi}{2}\alpha}}{\pi}, \quad M = \frac{\sqrt{2\sin\frac{\pi}{2}\alpha}}{\pi}. \quad (2.12)$$

N and M are normalization factors and K_ν is the McDonald function of order ν ([GR]). The phase factors in $\xi_-^{(0)}$, $\xi_-^{(-1)}$ are irrelevant at this step. They have been introduced only to be consistent with the choice of an analytic basis for the deficiency subspaces of \dot{H}_α^* (see below).

It is well known (see e.g. [RSII],[BG]) that $\dot{h}_{\alpha,m}$ are s.a. for $m \neq 0, -1$, while $\dot{h}_{\alpha,0}$ and $\dot{h}_{\alpha,-1}$ have deficiency indices (1,1).

The adjoint operators of $\dot{h}_{\alpha,0}$ and $\dot{h}_{\alpha,-1}$ are defined by

$$\begin{aligned} D(\dot{h}_{\alpha,0}^*) &= \left\{ \xi \in L^2(R^+) \mid \xi, \xi' \in H_{loc}^1(R^+), \right. \\ &\quad \left. -\xi'' + (\alpha^2 - 1/4)r^{-2}\xi \in L^2(R^+) \right\}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} D(\dot{h}_{\alpha,-1}^*) &= \left\{ \xi \in L^2(R^+) \mid \xi, \xi' \in H_{loc}^1(R^+), \right. \\ &\quad \left. -\xi'' + [(1-\alpha)^2 - 1/4]r^{-2}\xi \in L^2(R^+) \right\}, \end{aligned} \quad (2.14)$$

$$\dot{h}_{\alpha,m}^*\xi = -\frac{d^2\xi}{dr^2} + \frac{(m+\alpha)^2 - 1/4}{r^2}\xi, \quad m = 0, -1. \quad (2.15)$$

Moreover (2.10) and (2.11) are the unique solutions (up to a constant factor) of $\dot{h}_{\alpha,0}^*\zeta = \pm i\zeta$ and $\dot{h}_{\alpha,-1}^*\zeta = \pm i\zeta$ respectively.

From (2.4),(2.15) we obtain

$$\dot{H}_\alpha^* = \bigoplus_{m=-\infty}^{+\infty} V^{-1}\dot{h}_{\alpha,m}^*V \otimes 1. \quad (2.16)$$

Using (2.16) we easily find that the equation $\dot{H}_\alpha^* \psi = \pm i \psi$ has two independent solutions

$$\psi_\pm^{(0)}(r) = \frac{r^{-1/2}}{\sqrt{2\pi}} \xi_\pm^{(0)}(r), \quad \psi_\pm^{(-1)}(r, \phi) = \frac{r^{-1/2}}{\sqrt{2\pi}} \xi_\pm^{(-1)}(r) e^{-i\phi}. \quad (2.17)$$

This means that \dot{H}_α has deficiency indices (2,2) and then a four-parameter family of s.a. extensions.

We take $\{\psi_\pm^{(0)}, \psi_\pm^{(-1)}\}$ as a basis for the two dimensional deficiency subspaces $\chi_\pm \equiv \ker(\dot{H}_\alpha^* \mp i)$ and denote by U any unitary map from χ_+ to χ_- . Applying the standard theory ([AG]), one finds that such extensions are explicitly given by

$$D(H_\alpha^U) = \{u \in L^2(R^2) \mid u = v + \psi_+ + U\psi_+\}, \quad (2.18)$$

$$H_\alpha^U u = \dot{H}_\alpha v + i\psi_+ - iU\psi_+, \quad (2.19)$$

where $v \in D(\dot{H}_\alpha)$, $\psi_+ = c_0 \psi_+^{(0)} + c_{-1} \psi_+^{(-1)}$, $c_0, c_{-1} \in \mathbb{C}$, $U\psi_+ = \tilde{c}_0 \psi_-^{(0)} + \tilde{c}_{-1} \psi_-^{(-1)}$, $\tilde{c}_j = \sum_{l=0,-1} \tilde{U}_{jl} c_l$, $j = 0, -1$, and \tilde{U} is a 2×2 unitary matrix which can be represented as

$$\tilde{U} = e^{i\eta} \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1. \quad (2.20)$$

For any possible choice of the parameters η , a , b we obtain a s.a. realization of the formal hamiltonian (1.1).

If we choose $\eta = 0$, $b = 0$, $a = -1$ the original extension H_α^{AB} studied by Aharonov-Bohm is obtained. In this case the domain of the hamiltonian consists of functions vanishing for $r \rightarrow 0$. The only parameter appearing in the hamiltonian is α , which correspond to a purely magnetic interaction (i.e. with no additional contact interaction in $r = 0$).

For $b = 0$ one has $a = e^{i\tau}$ and then the extensions are parametrized by τ, η . Such extensions have been studied recently in [GHKL],[GMS],[MT]. They correspond to a magnetic interaction plus a contact interaction acting in the s-wave and in the p-wave separately.

The choice $\eta = \pi + \tau$ (resp. $\eta = \pi - \tau$) eliminates the contact interaction acting in the p-wave (resp. in the s-wave).

For $b \neq 0$ one has extensions which don't commute with the angular momentum.

3. The Resolvent

In this Section we use Krein's method ([AG],[DG]) to compute the resolvent of the generic self-adjoint extension H_α^U .

The two basic ingredients are the knowledge of the resolvent of a particular extension of \dot{H}_α and the construction of an analytic basis for the deficiency subspaces.

In Sec. 2 we observed that for $\eta = 0$, $b = 0$, and $a = -1$, i.e. $U = -I$, one has $H_\alpha^U = H_\alpha^{AB}$, where the Aharonov-Bohm extension H_α^{AB} is an exactly solvable hamiltonian.

Its spectrum is $[0, +\infty)$, it is purely absolutely continuous, and the generalized eigenfunctions are

$$\Psi_\alpha^{AB}(r, \phi, k, \theta) = \sum_{m=-\infty}^{+\infty} i^{|m|} e^{im(\phi-\theta)} e^{i\frac{\pi}{2}(|m|-|m+\alpha|)} J_{|m+\alpha|}(kr) \quad (3.1)$$

(see e.g. [R]). J_ν is the Bessel function of the first kind and of order ν .

From (3.1) one easily obtains the explicit expression for the resolvent of H_α^{AB} .

$$\begin{aligned} [\mathcal{R}_\alpha^{AB}(k)f](r, \phi) &\equiv \left[(H_\alpha^{AB} - k^2)^{-1} f \right](r, \phi) \\ &= \int_0^{+\infty} \rho d\rho \int_0^{2\pi} d\zeta \sum_{m=-\infty}^{+\infty} i \frac{e^{im(\phi-\zeta)}}{4} \\ &\quad \times J_{|m+\alpha|}(k(r \wedge \rho)) H_{|m+\alpha|}^{(1)}(k(r \vee \rho)) f(\rho, \zeta) \end{aligned} \quad (3.2)$$

where $Im k > 0$, $(x \wedge y)$ and $(x \vee y)$ are respectively the minimum and the maximum between x and y . $H_\nu^{(1)}$ is the Hankel function of the first kind and of order ν .

The construction of an analytic basis can be made using (3.2).

We observe that the following map, defined for $Im k > 0$

$$\begin{aligned} \mathcal{V}(k, e^{i\frac{\pi}{4}}) &\equiv (H_\alpha^{AB} - i)(H_\alpha^{AB} - k^2)^{-1} \\ &= 1 + (k^2 - i)\mathcal{R}_\alpha^{AB}(k) \end{aligned} \quad (3.3)$$

is obviously analytic in k and defines an isomorphism between χ_+ and $\chi(k) \equiv \text{Ker}(\dot{H}_\alpha^* - k^2)$.

Then the analytic basis in $\chi(k)$ is:

$$\{\psi_k^{(0)}, \psi_k^{(-1)}\} = \{\mathcal{V}(k, e^{i\frac{\pi}{4}}) \psi_+^{(0)}, \mathcal{V}(k, e^{i\frac{\pi}{4}}) \psi_+^{(-1)}\} \quad (3.4)$$

For $k^2 = -i$ in (3.4) we reobtain the basis $\{\psi_-^{(0)}, \psi_-^{(-1)}\}$ for χ_- .

Now the generalized Krein's formula ([AG], [DG]) yields

$$\begin{aligned} \mathcal{R}_\alpha^U(k) f &\equiv (H_\alpha^U - k^2)^{-1} f \\ &= \mathcal{R}_\alpha^{AB}(k) f + \sum_{j,l=0,-1} p(k)_{jl} (\psi_{-\bar{k}}^{(j)}, f) \psi_k^{(l)} \end{aligned} \quad (3.5)$$

Here the 2×2 matrix $p(k)$ is determined by

$$p(k) = \left(1 + (k^2 - i) p(e^{i\frac{\pi}{4}}) A(k, e^{i\frac{\pi}{4}})\right)^{-1} p(e^{i\frac{\pi}{4}}) \quad (3.6)$$

where $A(k_1, k_2)$ is a 2×2 complex matrix whose elements can be written as follows

$$\begin{aligned} A(k_1, k_2)_{jl} &\equiv (\psi_{-\bar{k}_1}^{(j)}, \psi_{k_2}^{(l)}) \\ &= \frac{1}{k_2^2 - k_1^2} \left[\frac{(-k_1^2)^\alpha - (-k_2^2)^\alpha}{\sin(\frac{\pi}{2}\alpha)} \delta_{j0} \delta_{jl} \right. \\ &\quad \left. + \frac{(-k_1^2)^{1-\alpha} - (-k_2^2)^{1-\alpha}}{\cos(\frac{\pi}{2}\alpha)} \delta_{jl} \delta_{-1l} \right] \quad j, l = 0, -1 \end{aligned} \quad (3.7)$$

$p(e^{i\frac{\pi}{4}})$ is a 2×2 matrix which can be computed following ([DG])

$$\begin{aligned} p(e^{i\frac{\pi}{4}}) &= \frac{i}{2} A(e^{i\frac{\pi}{4}}, e^{i\frac{3}{4}\pi})^{-1} \left(-I - \overline{U} \right) \\ &= -\frac{i}{2} \begin{pmatrix} 1 + e^{-i\eta} \bar{a} & -e^{-i\eta} b \\ e^{-i\eta} \bar{b} & 1 + e^{-i\eta} a \end{pmatrix} \end{aligned} \quad (3.8)$$

We have denoted in (3.8) by \overline{B} the matrix obtained from the matrix B taking the complex conjugate of each element.

It is easy to show that under the condition $Im\,k > 0$ the inverse matrix in (3.6) exists, except for (at most) two values of k , corresponding to the bound states of H_α^U ([DG]).

A straightforward computation shows that

$$\begin{aligned}
p_{00}(k) &= \frac{e^{-i\eta}}{2D} \left[\frac{a' + \cos \eta}{\cos\left(\frac{\pi}{2}\alpha\right)} \left((-k^2)^{1-\alpha} - (-i)^{1-\alpha} \right) - i(e^{i\eta} + \bar{a}) \right] \\
p_{0-1}(k) &= \frac{i e^{-i\eta}}{2D} b \\
p_{-10}(k) &= -\frac{i e^{-i\eta}}{2D} \bar{b} \\
p_{-1-1}(k) &= \frac{e^{-i\eta}}{2D} \left[\frac{a' + \cos \eta}{\sin\left(\frac{\pi}{2}\alpha\right)} \left((-k^2)^\alpha - (-i)^\alpha \right) - i(e^{i\eta} + a) \right] \quad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
D &= \det \left(1 + (k^2 - i) p(e^{i\frac{\pi}{4}}) A(k, e^{i\frac{\pi}{4}}) \right) \\
&= c_1(-k^2) + c_\alpha(-k^2)^\alpha + c_{1-\alpha}(-k^2)^{1-\alpha} + c_0 \quad (3.10)
\end{aligned}$$

with

$$\begin{aligned}
c_1 &= -\frac{e^{-i\eta}}{\sin(\pi\alpha)} (a' + \cos \eta) \\
c_\alpha &= \frac{e^{-i\eta}}{\sin(\pi\alpha)} \left[a' \sin\left(\frac{\pi}{2}\alpha\right) + \sin\left(\frac{\pi}{2}\alpha - \eta\right) + a'' \cos\left(\frac{\pi}{2}\alpha\right) \right] \\
c_{1-\alpha} &= \frac{e^{-i\eta}}{\sin(\pi\alpha)} \left[a' \cos\left(\frac{\pi}{2}\alpha\right) + \cos\left(\frac{\pi}{2}\alpha + \eta\right) - a'' \sin\left(\frac{\pi}{2}\alpha\right) \right] \\
c_0 &= \frac{e^{-i\eta}}{\sin(\pi\alpha)} (\sin \eta - a'' \cos(\pi\alpha) - a' \sin(\pi\alpha)) \quad (3.11)
\end{aligned}$$

and

$$\begin{aligned}
a' &= Re\,a \\
a'' &= Im\,a \quad (3.12)
\end{aligned}$$

Due to the structure of the resolvent $\mathcal{R}_\alpha^U(k)$ as given by (3.5), one can easily obtain information on the spectral properties of H_α^U (cfr. the analogous situation for ordinary point interactions in [AGH-KH]).

In particular one has

$$\sigma_{ac}(H_\alpha^U) = \sigma_{ac}(H_\alpha^{AB}) = [0, +\infty) \quad \sigma_{sing}(H_\alpha^U) = \emptyset \quad (3.13)$$

Moreover the pure point spectrum consists of (at most) two negative eigenvalues determined as the solutions of

$$\det \left(1 + (k^2 - i) p(e^{i\frac{\pi}{4}}) A(k, e^{i\frac{\pi}{4}}) \right) = 0 \quad (3.14)$$

Concerning the scattering, one knows (see e.g. [R]) that the wave operators associated to the pair $(H_\alpha^{AB}, -\Delta)$ exist and are complete in the strongest sense

$$\text{Ran} \left(\Omega_\alpha^{AB} \right)_+ = \text{Ran} \left(\Omega_\alpha^{AB} \right)_- = \mathcal{H}_{bound\ states}^\perp \quad (3.15)$$

and the eigenstates have negative energy.

Using the chain rule for the wave operators and the Kato-Birman theorem ([RS III, SIM]), we conclude that also the wave operators for the pair $(H_\alpha^U, -\Delta)$ exist and are complete in the strongest sense.

We observe that eq. (3.14) is easily analyzed in the special case $\eta = 0$, $a = 0$, $b = e^{i\gamma}$, corresponding to the simplest, non-rotationally invariant hamiltonian.

In fact for $k^2 = -|E|$, eq. (3.14) now reads

$$-|E| + \sin \left(\frac{\pi}{2} \alpha \right) |E|^\alpha + \cos \left(\frac{\pi}{2} \alpha \right) |E|^{1-\alpha} = 0 \quad (3.16)$$

It is easy to verify that (3.16) has two solutions: $|E| = 0$, corresponding to a zero-energy resonance, and $|E| = E_1(\alpha)$ corresponding to a non rotationally invariant bound state.

Finally, one can also note that for the rotationally-invariant case ($b = 0$, $a = e^{i\tau}$) equation (3.14) becomes

$$\left[(-k^2)^{1-\alpha} \cos \omega - \sin \left(\frac{\pi}{2} \alpha - \omega \right) \right] \left[(-k^2)^\alpha \cos \beta - \cos \left(\beta + \frac{\pi}{2} \alpha \right) \right] = 0 \quad (3.17)$$

where $\omega = \frac{\eta-\tau}{2}$ and $\beta = \frac{\eta+\tau}{2}$.

From this factorization one obtains the equations for the bound states in the p- and s-wave respectively (cfr. [A], [MT]).

4. Stationary Scattering Theory

Here we use the above results to describe the stationary scattering theory for the pair $(H_\alpha^U, -\Delta)$.

In particular we compute the generalized eigenfunctions and the scattering amplitude for H_α^U .

A complete (time-dependent) analysis of the scattering for the couple $(H_\alpha^U, -\Delta)$ requires further technical work and will be discussed in a forthcoming paper. Denote by $R_\alpha^U(k; \rho, \theta, r, \phi)$, the integral kernel of the resolvent of H_α^U .

Then the generalized eigenfunctions $\Psi_\alpha^U(k, \theta, r, \phi)$, with $k \geq 0$, of H_α^U are obtained through the standard limit procedure (see e.g. [AG], [AGH-KH])

$$\begin{aligned}
\Psi_\alpha^U(k, \theta, r, \phi) &= \lim_{\varepsilon \downarrow 0} \lim_{\rho \rightarrow +\infty} \frac{4}{i H_0^{(1)}((k + i\varepsilon)\rho)} R_\alpha^U(k + i\varepsilon; \rho, \theta + \pi, r, \phi) \quad (4.1) \\
&= \Psi_\alpha^{AB}(k, \theta, r, \phi) \\
&\quad + 2i^{\alpha+1} \cos\left(\frac{\pi}{2}\alpha\right) p_{00}(k) (-k^2)^\alpha H_\alpha^{(1)}(kr) \\
&\quad - i^\alpha \sqrt{2 \sin(\pi\alpha)} e^{-i\pi(1-2\alpha)} p_{-10}(k) k H_\alpha^{(1)}(kr) e^{i\theta} \\
&\quad + i^{1-\alpha} \sqrt{2 \sin(\pi\alpha)} e^{i\pi(1-2\alpha)} p_{0-1}(k) k H_{1-\alpha}^{(1)}(kr) e^{-i\phi} \\
&\quad - 2i^{2-\alpha} \sin\left(\frac{\pi}{2}\alpha\right) p_{-1-1}(k) (-k^2)^{1-\alpha} H_{1-\alpha}^{(1)}(kr) e^{-i(\phi-\theta)} \quad (4.2)
\end{aligned}$$

Finally we determine the scattering amplitude $f_\alpha^U(k, \theta, \phi)$ from the asymptotic behaviour of $\Psi_\alpha^U(k, \theta, r, \phi)$ for large r , i.e.

$$\Psi_\alpha^U(k, \theta, r, \phi) \xrightarrow{r \rightarrow +\infty} e^{ikr \cos(\phi-\theta)} + f_\alpha^U(k, \theta, \phi) \frac{e^{ikr}}{\sqrt{r}} \quad (4.3)$$

where

$$\begin{aligned}
f_{\alpha}^U(k, \theta, \phi) &= f_{\alpha}^{AB}(k, \theta, \phi) \\
&+ 4 \sqrt{\frac{2i\pi}{k}} \cos\left(\frac{\pi}{2}\alpha\right) p_{00}(k) (-k^2)^{\alpha} \\
&- 2 \sqrt{\frac{2\pi}{ik}} \sqrt{2 \sin(\pi\alpha)} e^{-i\pi(1-2\alpha)} p_{-10}(k) k e^{i\theta} \\
&+ 2 \sqrt{\frac{2\pi}{ik}} \sqrt{2 \sin(\pi\alpha)} e^{i\pi(1-2\alpha)} p_{0-1}(k) k e^{-i\phi} \\
&- 4 \sqrt{\frac{2i\pi}{k}} \sin\left(\frac{\pi}{2}\alpha\right) p_{-1-1}(k) (-k^2)^{1-\alpha} e^{-i(\phi-\theta)} \quad (4.4)
\end{aligned}$$

We have denoted by f_{α}^{AB} the scattering amplitude associated to H_{α}^{AB} explicitly given by (see e.g. [R]):

$$f_{\alpha}^{AB} = \sqrt{\frac{2\pi}{ik}} \left[\delta(\phi - \theta) (\cos(\pi\alpha) - 1) + i \frac{\sin(\pi\alpha)}{\pi} P \frac{1}{e^{-i(\phi-\theta)} - 1} \right] \quad (4.5)$$

The above formula for the scattering amplitude seems to be of interest. For $b = 0$ it shows that the scattering process can be reduced in each subspace of fixed angular momentum.

On the other hand, for $b \neq 0$, i.e., for any extension which is not rotationally invariant, one obtains from (4.4) a non vanishing and computable probability for the scattering process: incoming particle with angular momentum zero (resp. minus one) \longrightarrow outgoing particle with angular momentum minus one (resp. zero); the probability is proportional to $|p_{0-1}(k)|^2$ (resp. $|p_{-10}(k)|^2$). The lack of conservation of angular momentum for such hamiltonians can therefore be directly checked on a physical observable quantity, the scattering cross section for a given scattering process.

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During the final draft of the paper we became aware that L. Dabrowski and P. Stovicek were concluding a preprint on the same subject ([DS]).

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